

# Some Inequalities of Geometric Means in Grassmannians

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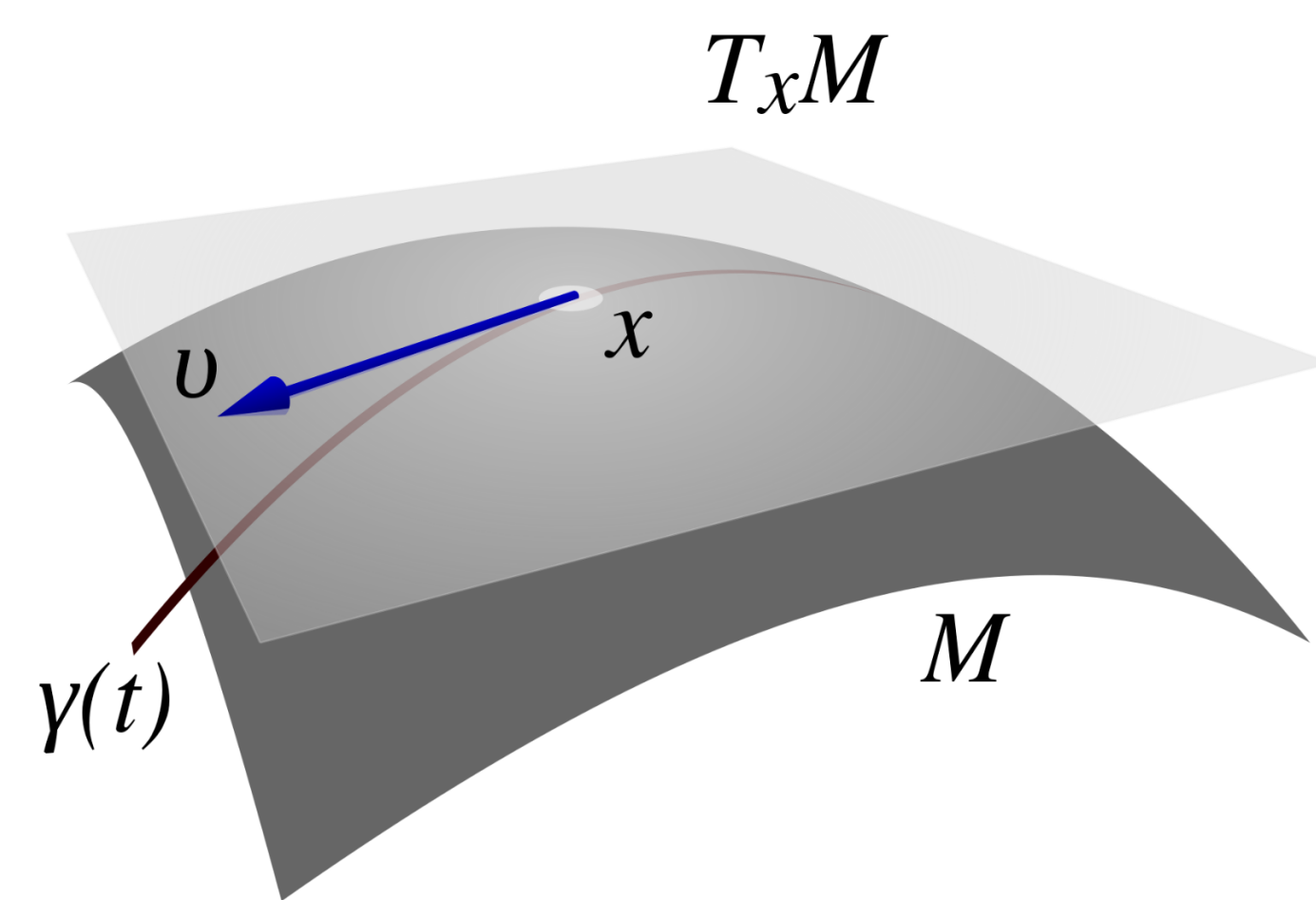


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## Abstract

We obtain some inequalities for the geometric means of two elements in the Grassmannians, for example, semi-parallelgram law, geodesic triangle inequalities, majorization for geodesic triangle and their related inequalities. These inequalities reflect the elliptic geometry of the Grassmannians as Riemannian manifolds.

## Introduction



The smooth manifold  $M$  endowed with a Riemannian metric is a Riemannian manifold. The followings are important concepts of a Riemannian manifold.

- **Riemannian metric:** every smooth manifold carries a Riemannian metric.
- **Tangent space:** the tangent space of  $x$  in  $M$  denoted by  $T_x M$ .
- **Geodesic:** the shortest distance between any two points in  $M$ , and denoted by

$$\gamma(t) : [0, 1] \rightarrow M$$

s.t.  $\gamma(0) = x, \dot{\gamma}(0) = v$ .

- **$t$ -geometric mean:**  $A \sharp_t B = \gamma(t)$  when  $\gamma(0) = A$  and  $\gamma(1) = B$ .

Grassmannian  $\text{Gr}_{n,k}(\mathbb{F})$  is one Riemannian manifold that parameterizes the set of all  $k$ -dimensional linear subspaces of an  $n$ -dimensional vector space over a field  $\mathbb{F}$ .

## Basic knowledge of Grassmannians

- The orthogonal projector models of Grassmannians are as follow. The set of orthogonal projection matrices in  $\mathbb{C}_{n \times n}$  of rank  $k$  is denoted by

$$\text{Gr}_{n,k}(\mathbb{C}) = \{P \in \mathbb{C}_{n \times n} : P^2 = P, P^* = P, \text{rank } P = k\} \subset \mathbb{H}_n,$$

where  $\mathbb{H}_n$  is the space of  $n \times n$  Hermitian matrices. Its real analog is

$$\text{Gr}_{n,k}(\mathbb{R}) = \{P \in \mathbb{R}_{n \times n} : P^2 = P, P^T = P, \text{rank } P = k\} \subset \mathbb{S}_n,$$

where  $\mathbb{S}_n$  is the space of  $n \times n$  real symmetric matrices.

- The tangent space  $T_P \text{Gr}_{n,k}(\mathbb{C})$  of  $\text{Gr}_{n,k}(\mathbb{C})$  at the point  $P$  is

$$T_P \text{Gr}_{n,k}(\mathbb{C}) = \{\Delta = [\Omega, P] = \Omega(I - 2P) : \Omega \in \mathfrak{u}_p(n)\},$$

and  $\mathfrak{u}_p(n) := \{\Omega \in \mathfrak{u}(n) : \Omega = \Omega P + P \Omega\}$ , where  $\mathfrak{u}(n)$  is the set of  $n \times n$  skew Hermitian matrices.

- The geodesic and  $t$ -geometric mean  $\gamma(t)$  joining  $P$  and  $Q$  can be expressed as

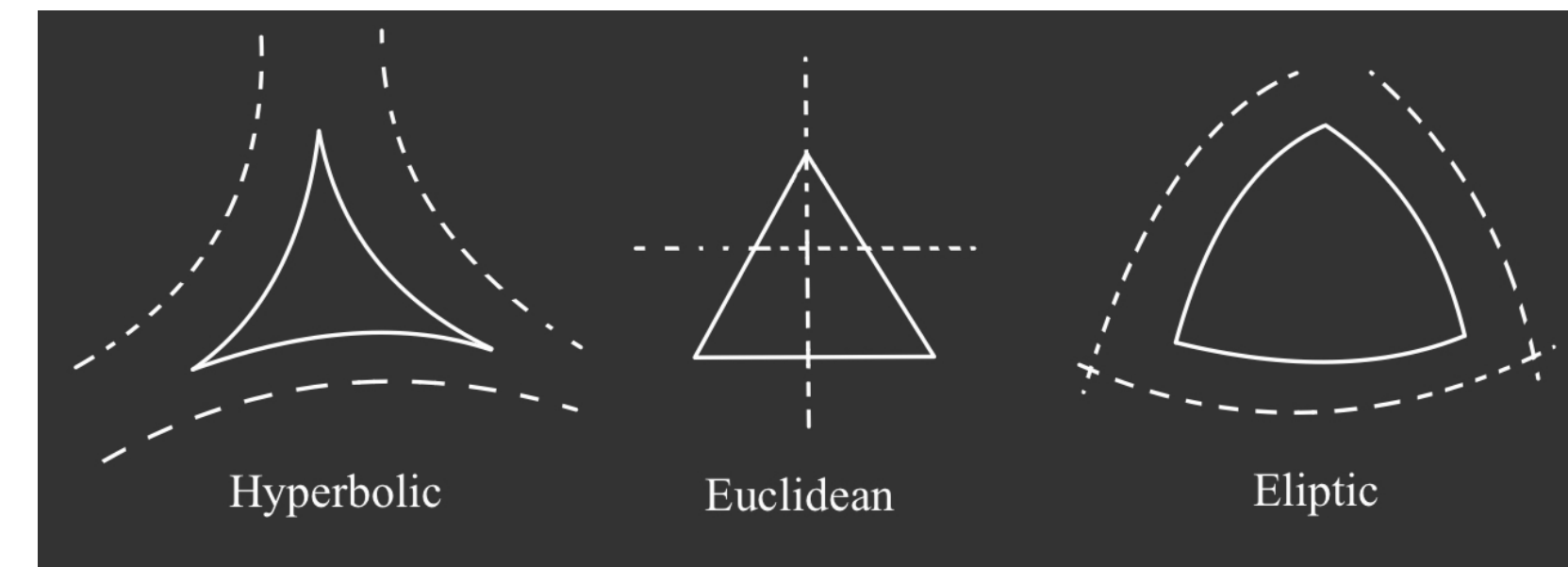
$$\gamma(t) = P \sharp_t Q = \text{Exp}_P(t \text{Exp}_P^{-1} Q), \quad t \in [0, 1]$$

in particular,  $\gamma_{\frac{1}{2}} = P \sharp Q \gamma(\frac{1}{2}) = P \sharp Q$  and

$$d(P, R) = \langle \text{Exp}_P^{-1} Q, \text{Exp}_P^{-1} Q \rangle_P^{1/2}$$

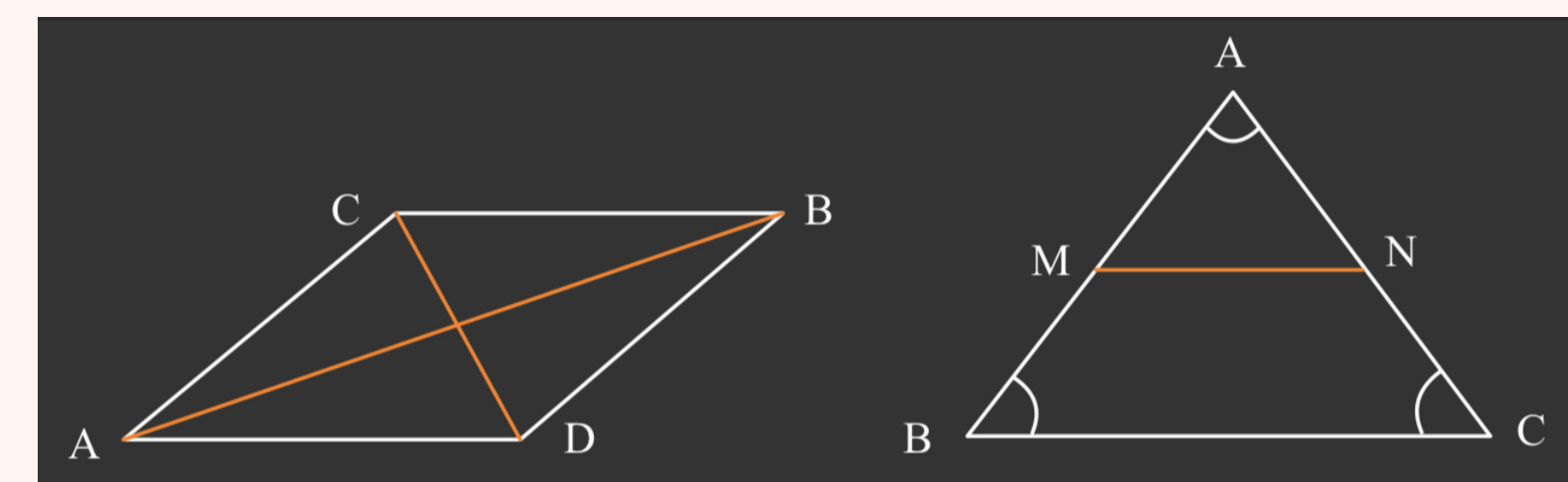
Note that the exponential map at  $P$ , denoted by  $\text{Exp}_P$ , is the diffeomorphism from a neighborhood of  $0 \in T_P \text{Gr}_{n,k}(\mathbb{F})$  to a neighborhood of  $P$ .

## Hyperbolic, Euclidean, Elliptic Geometry



In hyperbolic (elliptic, resp.) geometry, two geodesics emitting from the same point move away from each other faster (slower, resp.) than they were in an Euclidean space. Grassmannians have the elliptic geometry.

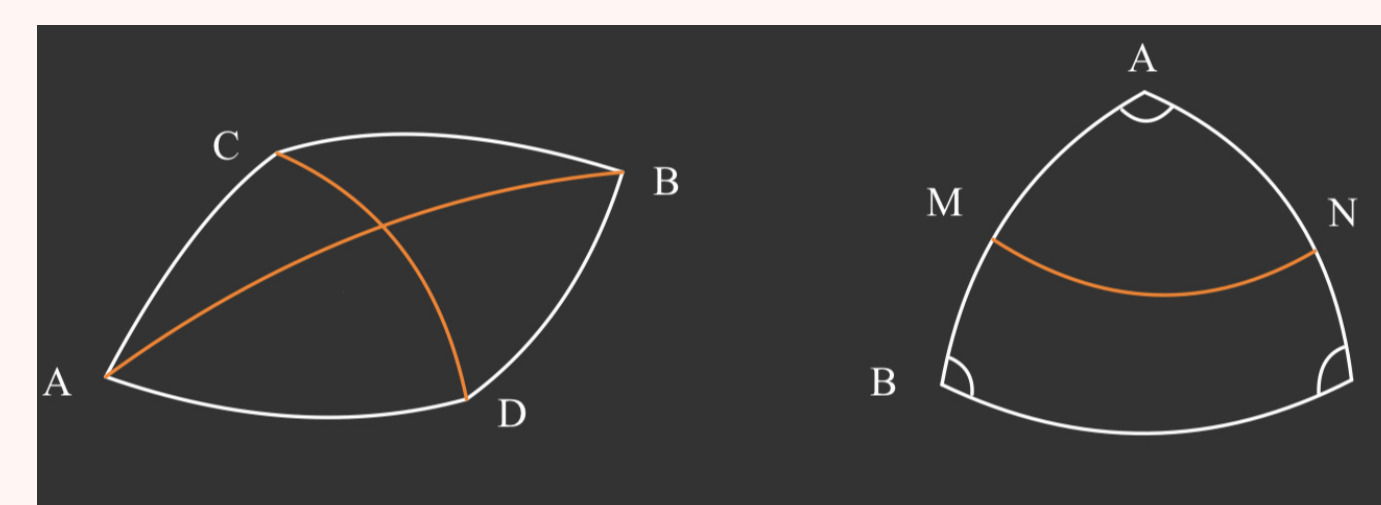
## Motivation and Challenge



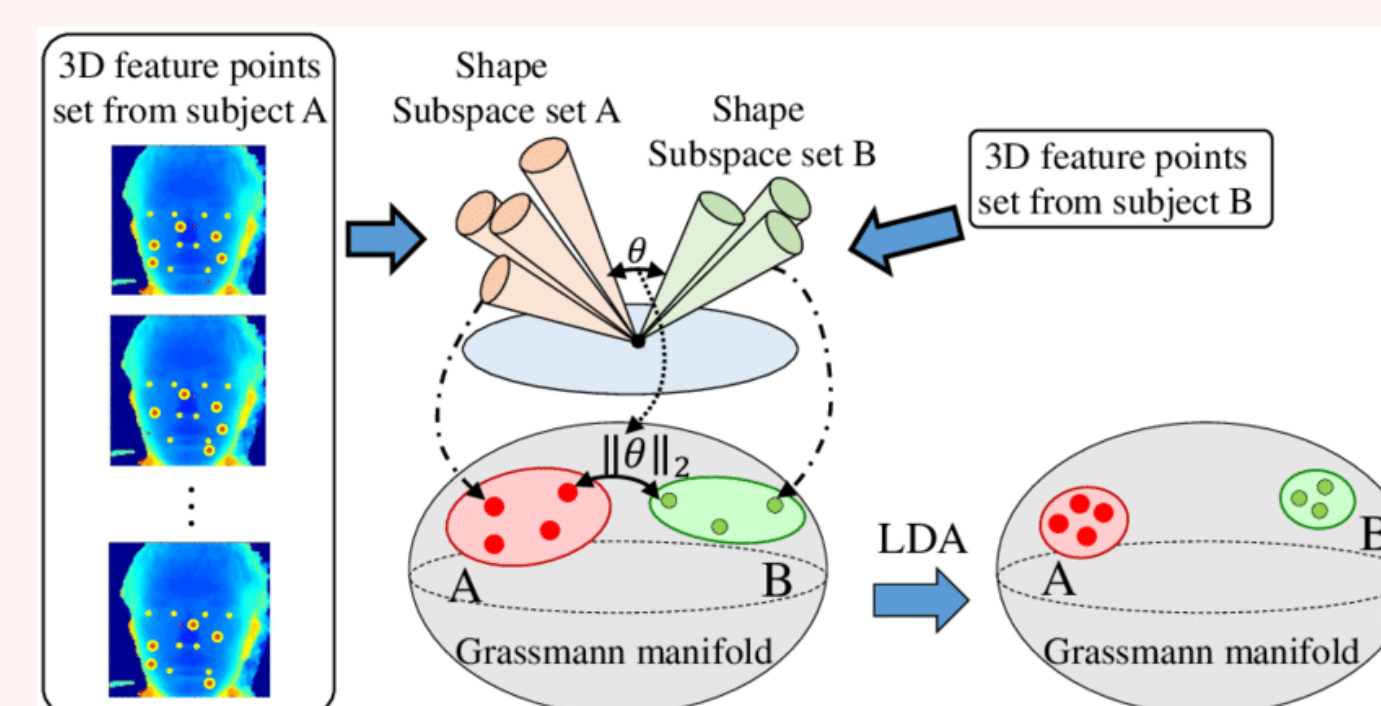
In Euclidean space, there are many famous law, such as

- The Parallelogram Law:  $d^2(A, B) + d^2(C, D) = d^2(A, C) + d^2(B, C) + d^2(A, D) + d^2(B, D)$
- The Cosine Law:  $d^2(B, A) = d^2(C, A) + d^2(C, B) - 2d(C, A)d(C, B) \cos \gamma$
- If  $M$  and  $N$  are midpoints of the sides  $AB$  and  $AC$  respectively then  $\frac{d(B, C)}{2} = d(M, N)$

What about in Grassmannians?



Moreover, Grassmannians have a lot of applications such that Recognition of Digital Images of the Human Face.



However, there are some challenges to get some inequalities in Grassmannians.

- We don't know the explicit expression of the geodesic of Grassmannians.
- It is not true that any two points in Grassmannians can be joined by a unique geodesic. So the distance between two points in Grassmannians is not unique.

## Methods

- Apply the general form of the geodesic and the relation between the geodesic of grassmannians with the explicit geodesic of other Riemannian manifolds.
- Find a metric ball  $\mathcal{B}$  of radius less than  $\rho_{\mathbb{F}}$  in  $\text{Gr}_{n,k}(\mathbb{F})$  is strongly convex, i.e., for any  $p, q \in \mathcal{B}$  there is a unique minimizing geodesic from  $p$  to  $q$  in  $\text{Gr}_{n,k}(\mathbb{F})$ .

Therefore, all results we get are local inequalities in  $\mathcal{B}$  of  $\text{Gr}_{n,k}(\mathbb{F})$ .

## Results

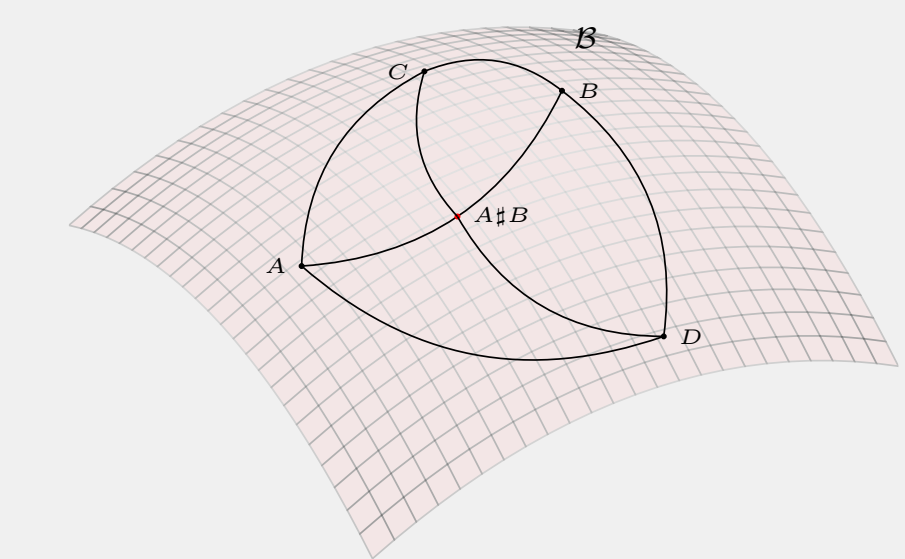
### Inequalities of distances on geometric means in Grassmannians

Let  $A, B, C \in \text{Gr}_{n,k}(\mathbb{F})$  in a metric ball  $\mathcal{B}$  of radius less than  $\rho_{\mathbb{F}}$ . We have the following inequalities.

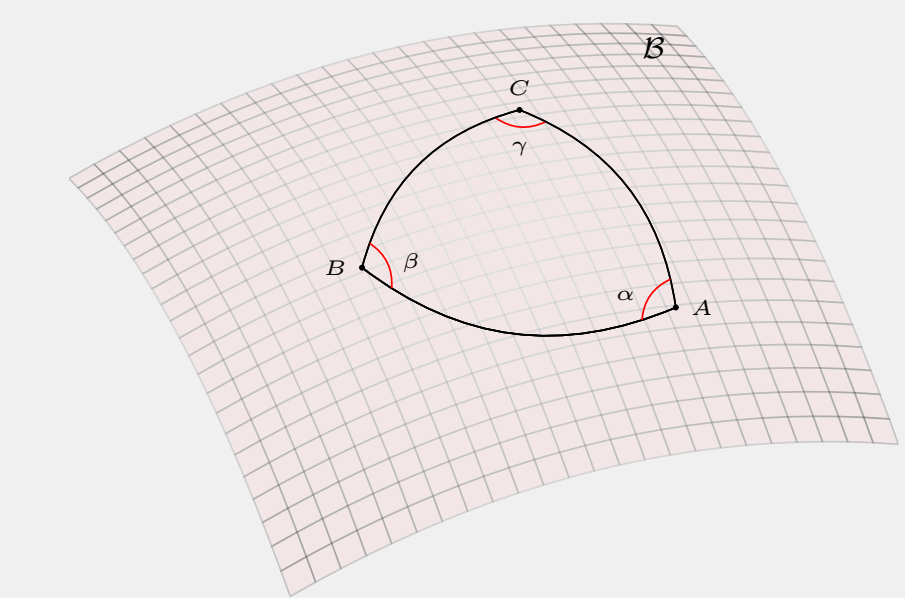
$$d^2(A \sharp B, C) \geq \frac{d^2(A, C) + d^2(B, C)}{2} - \frac{d^2(A, B)}{4}.$$

If  $D$  in  $\mathcal{B}$  exists such that the  $A \sharp B$  is the midpoint of the geodesic from  $C$  to  $D$ , then we have

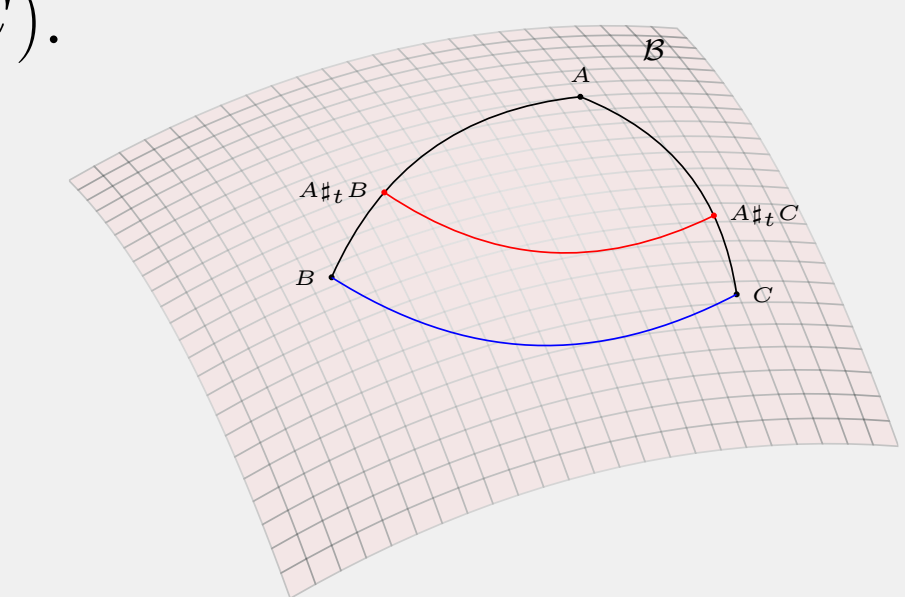
$$d^2(C, D) + d^2(A, B) \geq d^2(A, C) + d^2(B, C) + d^2(A, D) + d^2(B, D).$$



$$d^2(B, A) \leq d^2(C, A) + d^2(C, B) - 2d(C, A)d(C, B) \cos \gamma, \quad \alpha + \beta + \gamma \geq \pi.$$

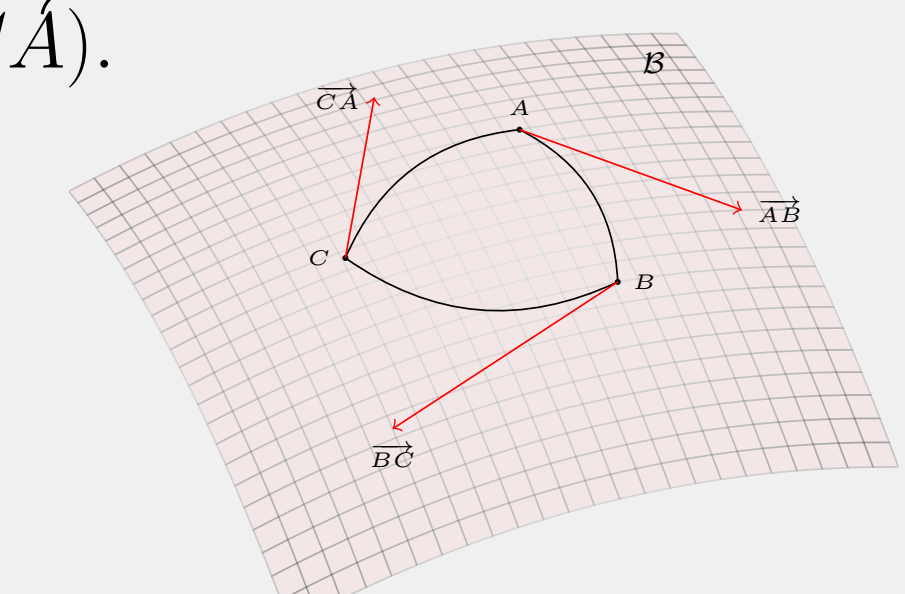


$$td(B, C) \leq d(A \sharp_t B, A \sharp_t C).$$



### Inequalities of tangent vectors on geometric means in Grassmannians

$$\sigma(\overrightarrow{AB}) \prec_w \sigma(\overrightarrow{BC}) + \sigma(\overrightarrow{CA}).$$



$$t\sigma(\overrightarrow{BC}) \prec_w \sigma(\overrightarrow{(A \sharp_t B)(A \sharp_t C)}).$$

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